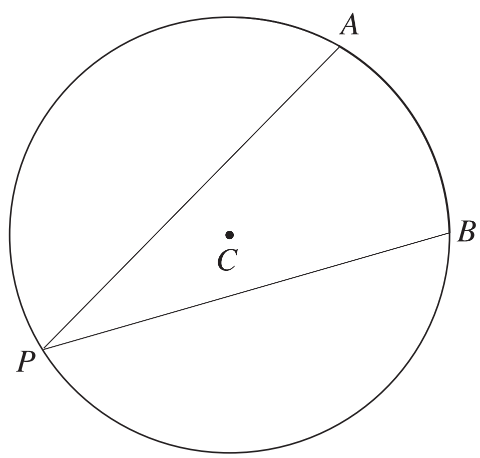
Exam E (Part I) **Name**

Please Note: For full credit, you will need to present your arguments *clearly, neatly, and in a well organized way* (as you would in an important memo to the boss). Please do your preliminary work on scratch paper. Your solutions on your test should be *complete and in final form.*

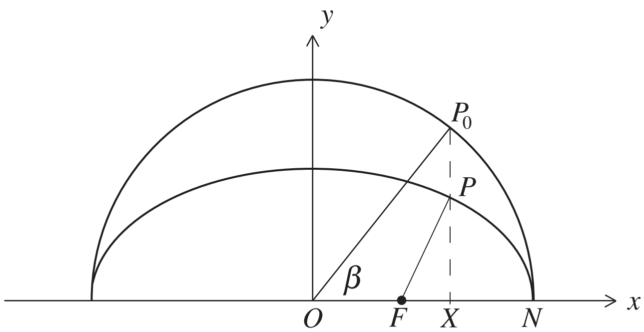
1. Take a circle of radius 3. Consider the points A and B on the circle as well as the arc that they determine. Let P be a point on the circle outside the arc and let $\angle APB = \frac{\pi}{7}$. Compute the length of the arc AB. Compute the area of the sector that the arc and the center C of the circle determine.



2. The distance from the focus F of a parabola to its directrix is 3. The parabola is cut parallel to the directrix at a distance of 7 from the directrix. Determine the area of the parabolic section.

3. Use Leibniz's tangent method to compute the slope of the tangent to the curve $y^2 = 4x^2 - 1$ at any point $P = (x, y)$. Make use of your answer (but not the chain rule) to show that the derivative of the function $f(x) = \sqrt{4x^2 - 1}$ is $f'(x) = \frac{4x}{\sqrt{4x^2-1}}$.

4. The figure below shows a coordinate system along with the upper half of a circle of radius 5 and the upper half of an ellipse with semimajor axis 5. The origin O is the center of both the full circle and the full ellipse. The point $F = (2, 0)$ is a focus of the ellipse. The point P_0 is on the circle, the point P is on the ellipse, and the dotted segment P_0PX is perpendicular to the x-axis. The y-coordinate of P_0 is 4.

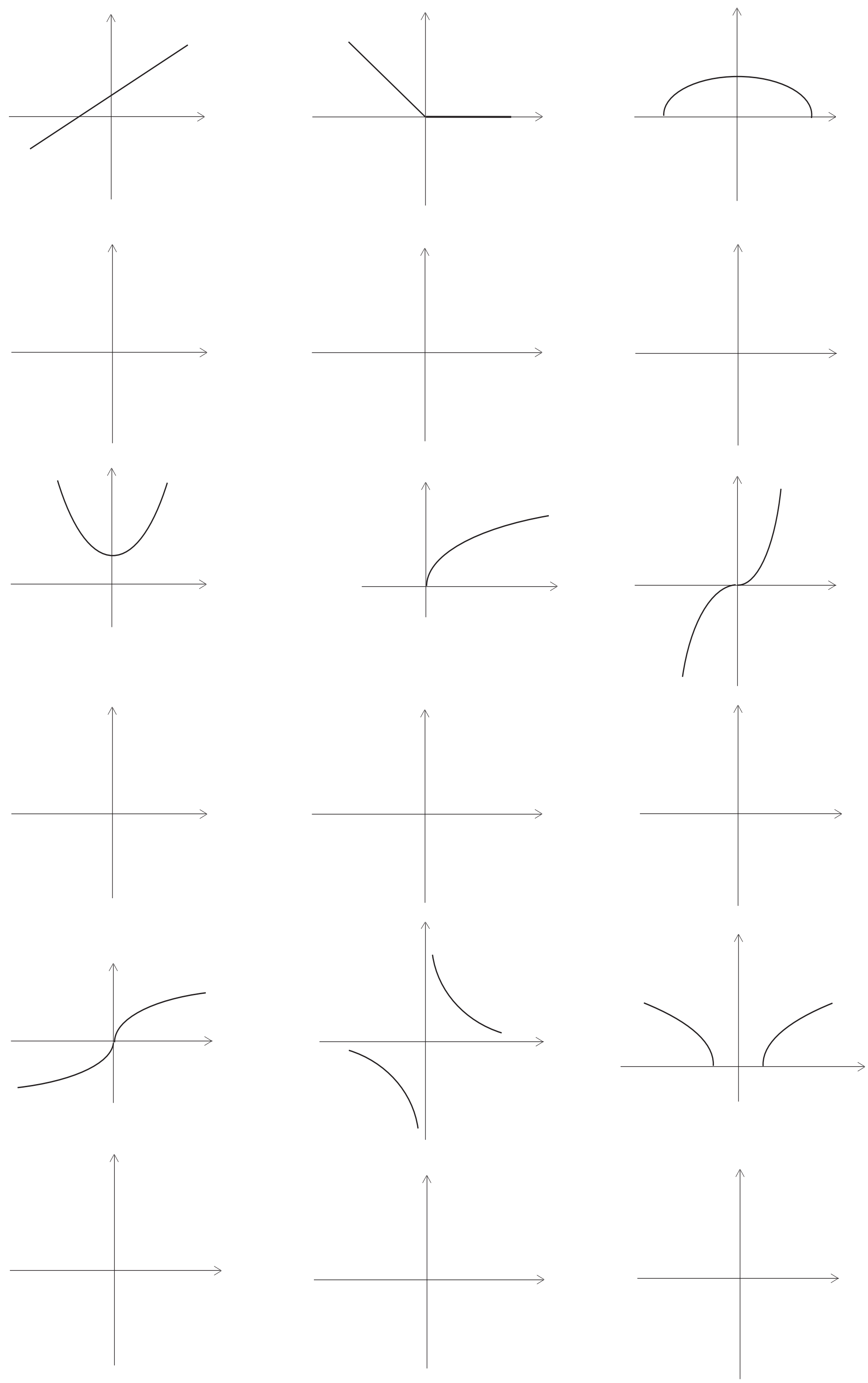


i. In the space above, compute the semiminor axis of the ellipse and write down an equation for the ellipse. (Just write it down, don't derive it.) Determine the y-coordinate of the point P.

ii. Compute the x-coordinate of the point X and the y-coordinate of P. Finally, compute the angle β in radians. (Use the "inverse sine" button of a calculator.)

iii. Compute the area of the circular section NXP_0 and then that of the elliptical sector NFP.

5. Each of the graphs sketched below is the graph of a function. Beside each graph, write down a specific function $f(x)$ whose graph looks like the one provided. Next, use the coordinate system below each graph to carefully sketch a curve that looks like the graph of the derivative of the function.



6. Consider the function $g(x) = 9 - x^2$ with $0 \leq x \leq 2$. Insert the points

$$0 \leq 0.3 \leq 0.5 \leq 0.8 \leq 1.2 \leq 1.5 \leq 1.7 \leq 1.9 \leq 2$$

on the x-axis between 0 and 2.

i. Compute the sum $g(x) \cdot dx$ that this set of points determines. Do so with three decimal place accuracy.

ii. Compute $\int_0^2 g(x)\,dx$ by using the Fundamental Theorem. Do so with three decimal place accuracy.

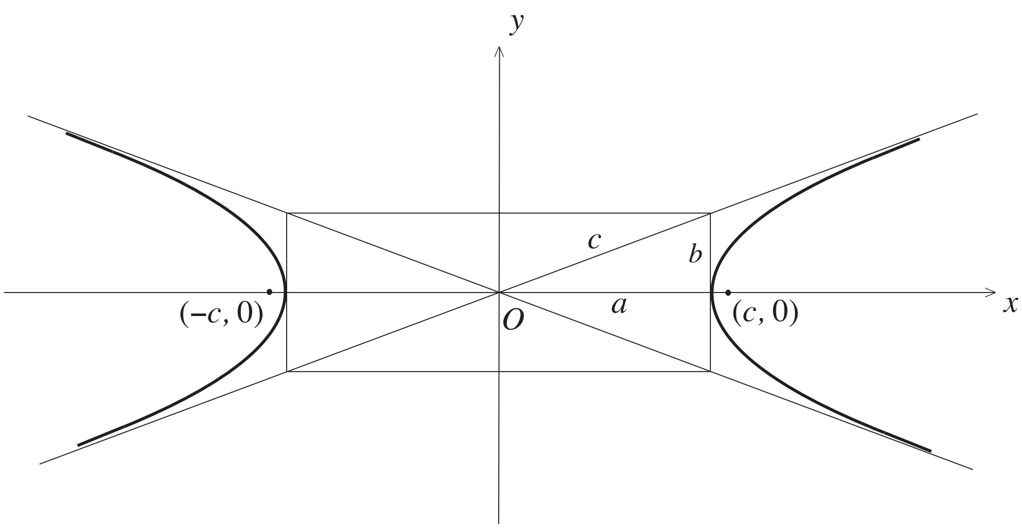
iii. What accounts for the difference between the answers in (i) and (ii)? What changes would you have to make to get the answer in (i) to be the same as that in (ii)? (Don't carry the changes out, just describe them.)

7. Consider the function $f(x) = 2x^3 + 3x^{\frac{1}{2}}$.

i. For any $x > 0$ find the area under the graph of $f(x)$ over the interval $[0, x]$ in terms of x.

ii. Determine the volume of the solid obtained by revolving the graph of $f(x)$ once around the x-axis.

The graph of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is sketched below. It is to a great extent determined by the rectangle at the center. The focal points of the hyperbola are $(-c, 0)$ and $(e, 0)$, where $c = \sqrt{a^2 + b^2}$. Its eccentricity is $\varepsilon = \frac{c}{a}$.

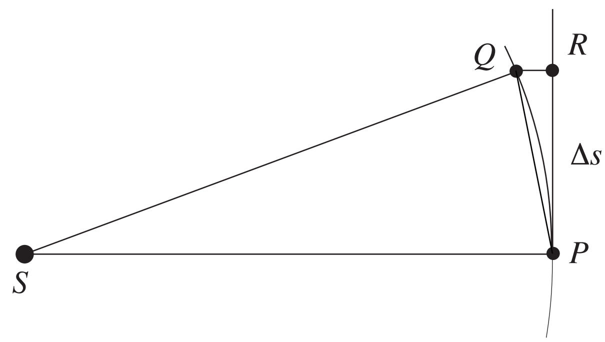


8. Consider the function $f(x) = \sqrt{4x^2 - 1}$. Show that its graph is the upper part of a hyperbola of the form discussed above.

i. Determine the focal points, the eccentricity, and the equations of the asymptotes. Compute the latus rectum $\frac{2b^2}{a}$ and identify the latus rectum on the graph above.

ii. Express the volume obtained by revolving the region under the hyperbola and above the segment $\frac{1}{2} \leq x \leq 4$ once around the x-axis as a definite integral.

9. The diagram below depicts a planet in its elliptical orbit. The semimajor and semiminor axes are a and b, the eccentricity is ε, and T is the period. The planet is in perihelion at P and at Q exactly Δt later. The point R is chosen so that QR is parallel to SP and Δs is the distance PR.



the position Q of the planet a short time Δt later. The segment SQQ' is a straight line. The segment PQ' has length Δs and lies on the tangent of the orbit at P.

i. With the aid of a diagram explain succinctly why Kepler's second law implies that the planet attains its maximum speed $v_{\max}$ at perihelion.

ii. Provide an expression for the precise value of the *average velocity* v_{av} of the motion of the planet from P to Q. Provide an approximation of v_{av} by using Δs.

iii. Use both the exact value $\kappa = \frac{ab\pi}{T}$ for Kepler's constant and an approximation for κ that arises from the given diagram to verify that the average velocity satisfies $v_{\text{av}} \approx \frac{2ab\pi}{(a-e)T}$.

iv. What two things happen to the approximation of (iii) when Δt is pushed to zero that result in the conclusion $v_{\max} = \frac{2ab\pi}{(a-e)T}$?

Formulas and Facts:

Area of circular sector equals: $\frac{1}{2}\theta r^2$
Archimedes's theorem: Area of parabolic section = $\frac{4}{3}$ × Area of inscribed triangle.

Cavalieri's principle: Consider two regions in the plane and let C and D be their areas. Let c_x and d_x be the respective cross-sections of the two regions for all x on some coordinate axis with $a \leq x \leq b$. If $c_x = kd_x$ for all x and some constant k, then $C = kD$.

$F_P = \frac{8\kappa^2 m}{L}\frac{1}{r_P^2}$ $\qquad F = G\frac{Mm}{r^2}$ $\qquad \frac{a^3}{T^2} = \frac{GM}{4\pi^2}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\qquad e^2 = a^2 - b^2$ $\qquad \varepsilon = \frac{e}{a}$ $\qquad V = \int_a^b \pi f(x)^2\,dx$